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Dual representations of GL_∞ and decomposition of Fock spaces

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Abstract. In this paper we study left and right 'regular representations' of $G = GL_\infty$, an infinite-dimensional analogue of the classical Lie group $GL(n, \mathbb{C})$ with entries indexed by the integers, on a space $\mathcal{F}$ which we take to be the projective limit of Bargmann–Fock–Segal spaces. We decompose $\mathcal{F}$ into an orthogonal direct sum $\mathcal{F} = \sum \hat{\oplus} \mathcal{I}^{(\omega)}$ where the hat indicates the closure of the algebraic direct sum, and where each $\mathcal{I}^{(\omega)}$ is irreducible with respect to the joint left and right action, and the isotypic component of the left or right actions with double signatures of the type $(\omega|\omega)$. For each ω, $\mathcal{I}^{(\omega)}$ is the subspace generated by the joint action on a highest weight vector $\Delta^{(\omega)}$ with highest weight ω. We start by defining the group GL_∞ and its Lie algebra $\mathfrak{gl}_\infty$, and discuss the 'infinite wedge space', F, defined by Kac, and its transpose F^t. We then define a kind of determinant function on GL_∞, prove several identities and properties of these determinants, and use these to produce intertwining maps from tensor products of F and F^t to $\mathcal{F}$, thereby obtaining irreducible subspaces and highest weight vectors. We further adapt these results to representations of GL_∞ on the inductive limit of Bargmann–Fock–Segal spaces $\mathfrak{F}$.

1. Introduction

The use of dual actions of pairs of groups has proven to be a useful tool in the theory of group representations by describing the irreducible representations of one group in terms of those of another.

The Schur–Weyl duality theorem (cf [17, 18]) classifies up to isomorphism all irreducible representations of the symmetric group of the order of n in terms of all irreducible rational representations of the general linear group of degree n. Racah, a physicist, discussed the relation between seniority and isospin, and his ideas were taken up by Moshinsky and Quesne [12], who then introduced the notion of *complimentary pairs* [13]. This was simultaneously generalized by Gross and Kunze in a series of papers starting with [1, 5] and further developed in [8, 16]. Most notably, Howe [6, 7], gave a general classification of *reductive dual pairs* of subgroups of a simplectic group, and this idea was further generalized by Klink and Ton-That in a series of papers (cf [11]) to that of dual pairs of *representations*, often of seemingly unrelated groups, typically on a Bargmann–Fock–Segal space (or just a 'Fock' space) of complex analytic functions of a matrix variable, where such a dual action is naturally implemented, and where there are applications to quantum physics [2].

In this paper we use this approach to study representations of GL_∞, an infinite-dimensional analogue of the classical Lie group $GL(n, \mathbb{C})$, which we take to be the inductive limit of groups isomorphic to $GL(n, \mathbb{C})$. We realize these representations on a space $\mathcal{F}$, the

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space of holomorphic functions on GL_∞, which we take to be the projective limit of Fock spaces isomorphic to $\mathcal{F}(\mathbb{C}^{(n\times n)})$. We start by defining GL_∞ and its Lie algebra $\mathfrak{gl}_\infty$, then construct the space $\mathcal{F}$, where we generalize the notion of orthogonality. We then define a joint action of $GL_\infty \times GL_\infty$ on $\mathcal{F}$ and decompose $\mathcal{F}$ into a multiplicity-free orthogonal direct sum

$$\sum \hat{\oplus} \mathcal{I}^{(\omega)}$$

where each $\mathcal{I}^{(\omega)}$ is an irreducible $GL_\infty \times GL_\infty$ module with highest weight ω, and also the isotypic component of the representations of GL_∞ with the signature ω. The hat indicates the closure of the algebraic direct sum, and the sum is taken over all highest weights (ω, ω).

We achieve this by adapting the results of representations of GL_∞ of the infinite wedge space, F, described in [9] and [10] and its 'transpose' F^t, on which all irreducible highest weight representations are known. We then define minor determinant functions of GL_∞ and prove some useful identities, particularly lemma 3.6. With these tools we are able to produce intertwining maps from tensor products of F and F^t to $\mathcal{F}$, decomposing $\mathcal{F}$ into copies of these known highest weight representations. We use these tools further to analyse the structure of these highest weight submodules. We further adapt these results to the more familiar case of representations GL_∞ on the inductive limits of Fock spaces.

2. Preliminaries

Much of this material is from [9] and [10]. For completeness it is included here, modified to suit our situation.

2.1. Infinite-dimensional vector spaces and matrices

Let

$$V = \sum_{i\in\mathcal{Z}} \oplus \mathbb{C} v_i$$

be a vector space over $\mathbb{C}$, the complex numbers, with fixed basis v_i. We may identify each v_i as a column vector with a 1 in the ith position and 0's elsewhere. Any vector v in V has a finite, but arbitrary number of non-zero coordinates. Similarly, we have the vector space

$$V^t = \sum_{j\in\mathcal{Z}} \oplus \mathbb{C} v_j^t$$

where v^t denotes the transpose of v.

The Lie algebra, $\mathfrak{gl}_\infty$, is defined as

$$\mathfrak{gl}_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} \in \mathbb{C} | a_{ij} = 0 \text{ for all but finitely many } i, j\}$$

with the Lie bracket just the ordinary matrix commutator product. $\mathfrak{gl}_\infty$ acts as usual by left matrix multiplication on the vector space V, and by right multiplication on V^t. If we denote by E_{ij} the matrix with 1 as the (i, j) entry and 0's elsewhere, then $\{E_{ij} | i, j \in \mathbb{Z}\}$ form a basis for $\mathfrak{gl}_\infty$.

We can regard $\mathfrak{gl}_\infty$ as the Lie algebra of the group GL_∞ where

$$GL_\infty = \{A = (A_{ij})_{i,j\in\mathbb{Z}} \in \mathbb{C} \ | A \text{ invertible and } A_{ij} - \delta_{ij} = 0 \text{ for all but finitely many } i, j\}$$

with the group operation being matrix multiplication, and the usual left action by left multiplication on V and the usual right action by right multiplication on V^t.

For $s, t \in \mathbb{Z}$, $s \leqslant t$, let $GL(s,t)$ be the subgroup of matrices acting on V which stabilizes v_j whenever $j < s$ or $j > t$. Then the collection of pairs $\{(s,t)\}_{s\leqslant t}$ forms a directed set with $(s,t) \leqslant (s',t')$ whenever $s' \leqslant s$ and $t \leqslant t'$ and we have that GL_∞ is the *direct limit* of the subgroups $GL(s,t)$, denoted somewhat redundantly as

$$GL_\infty = \varinjlim \bigcup_{s\leqslant t} GL(s,t).$$

Now for each $s \leqslant t$, $GL(s,t)$ is a Lie group and embeds in the obvious way into $GL(s',t')$, inheriting the subspace topology. We then give GL_∞ the *inductive limit topology*†, which we denote by

$$GL_\infty = \underset{\longrightarrow}{\mathrm{Ind}}\, GL(s,t).$$

This is the topology which makes the inclusion maps

$$GL(s,t) \overset{\mathrm{Inc}_{st}}{\hookrightarrow} GL_\infty$$

continuous for all $s \leqslant t$.

For any $s \leqslant t$ we also have the associated Lie algebra of $GL(s,t)$

$$\mathfrak{gl}_{st} = \{a \in \mathfrak{gl}_\infty | a_{ij} = 0 \text{ unless } s \leqslant i \leqslant t \text{ and } s \leqslant j \leqslant t\}$$

and in an analogous way we have that $\mathfrak{gl}_\infty$ is the inductive limit of the vector spaces $\mathfrak{gl}_{st}$,

$$\mathfrak{gl}_\infty = \underset{\longrightarrow}{\mathrm{Ind}} \bigcup_{s\leqslant t} \mathfrak{gl}_{st}.$$

2.2. The infinite wedge space $F(V)$

For any integer m, consider the sequence

$$m, m-1, \ldots, m-k, m-(k+1), \ldots$$

which we denote by $\underline{M}$. Let I_m (or just I if there is no ambiguity) be a finite shuffle of $\underline{M}$. That is, a mapping of finitely many elements of $\underline{M}$ to the integers $\mathbb{Z}$ that preserves the order;

$$I = i_m, i_{m-1}, i_{m-2}, \ldots$$

where

$$\begin{aligned} &\text{(i)} \qquad i_m > i_{m-1} > i_{m-2} > \cdots \\ &\text{(ii)} \qquad i_k = k + m \qquad \text{for } k \ll 0. \end{aligned} \tag{1}$$

Then for each $m \in \mathbb{Z}$ we construct the space $F^{(m)}(V)$ with basis elements of the form

$$v_I := v_{i_m} \wedge v_{i_{m-1}} \wedge v_{i_{m-2}} \wedge \ldots \tag{2}$$

where $\wedge$ denotes the exterior tensor product, and the indices satisfy (1). The basis elements of the form (2) are called *semi-infinite monomials* and are based on finite shuffles of the indices in the reference vector

$$v_{\underline{M}} = v_m \wedge v_{m-1} \wedge v_{m-2}.$$

We then set

$$F = F(V) = \sum_{m\in\mathbb{Z}} \oplus F^{(m)}(V)$$

† The terminology varies somewhat in the literature. Here we distinguish between the *direct limit*, an algebraic object, and the *inductive limit*, a topological object.

and in a completely analogous fashion we construct the space

$$F^t = F(V^t) = \sum_{m\in\mathcal{Z}} \oplus F^{(m)}(V^t)$$

where each space $F^{(m)}(V^t)$ has basis elements of the form

$$v_I^t := v_{i_m}^t \wedge v_{i_{m-1}}^t \wedge v_{i_{m-2}}^t \wedge \ldots$$

and indices also satisfying (1). A positive definite Hermitian form $\langle\cdot|\cdot\rangle$ is defined by declaring semi-infinite monomials to be orthonormal, i.e.

$$\langle v_I | v_J \rangle = \delta_J^I$$

linear in the first argument, and conjugate linear in the second.

Left multiplication by GL_∞ on V extends in the usual way to a left action, $\tilde{L}$, of GL_∞ on $F(V)$ by

$$\begin{aligned}\tilde{L}(A)(v_{I_m}) &= \tilde{L}(A)(v_{i_m} \wedge v_{i_{m-1}} \wedge v_{i_{m-2}} \wedge \ldots) \\ &= Av_{i_m} \wedge Av_{i_{m-1}} \wedge Av_{i_{m-2}} \wedge \ldots \end{aligned} \tag{3}$$

for each basis vector v_I of $F(V)$, and then extending by linearity. We will see that this action is continuous, so that (3) is a representation of the topological group GL_∞ on F in the usual sense. By formal differentiation of (3) we obtain a representation of $\mathfrak{gl}_\infty$ on F given by

$$\begin{aligned}\tilde{\ell}(a)(v_{I_m}) = \tilde{\ell}(a)(v_{i_m} \wedge v_{i_{m-1}} \wedge v_{i_{m-2}} \wedge \ldots)& \\ = av_{i_m} \wedge v_{i_{m-1}} \wedge v_{i_{m-2}} \wedge \ldots& \\ +v_{i_m} \wedge av_{i_{m-1}} \wedge v_{i_{m-2}} \wedge \ldots& \\ +v_{i_m} \wedge v_{i_{m-1}} \wedge av_{i_{m-2}} \wedge \ldots& \\ \vdots\ &\end{aligned}$$

We also have the right action, which we denote by $\tilde{R}$

$$\begin{aligned}\tilde{R}(A)(v_{I_m}^t) &= \tilde{R}(A)(v_{i_m}^t \wedge v_{i_{m-1}}^t \wedge v_{i_{m-2}}^t \wedge \ldots) \\ &= v_{i_m}^t A^t \wedge v_{i_{m-1}}^t A^t \wedge v_{i_{m-2}}^t A^t \wedge \ldots\end{aligned}$$

and a similar associated Lie algebra action, denoted by $\tilde{r}$.

By construction, $\tilde{\ell}(E_{ij})$ preserves the commutator product and maps each $F^{(m)}(V)$ onto itself, so that if $\tilde{\ell}_m$ is the restriction of $\tilde{\ell}$ to $F^{(m)}$ then

$$\tilde{\ell} = \sum_{m\in\mathbb{Z}} \oplus \tilde{\ell}_m.$$

Since for each m, $F^{(m)}(V)$ is the linear span of semi-infinite monomials of the form

$$\begin{aligned}\psi &= v_{i_m} \wedge \ldots \wedge v_{i_{m-k}} \wedge \ldots \\ &= \tilde{\ell}(E_{i_m,m}) \ldots \tilde{\ell}(E_{i_{m-k},m-k}) v_{\underline{M}}\end{aligned} \tag{4}$$

we have that each $F^{(m)}(V)$ is generated by the action of $\mathfrak{gl}_\infty$ on the reference vector $v_{\underline{M}} = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots$, and so each $F^{(m)}(V)$ is a $\mathfrak{gl}_\infty$ invariant subspace.

Proposition 2.1 (Kac). For each $m \in \mathbb{Z}$ the representation $\tilde{\ell}_m$ of $\mathfrak{gl}_\infty$ on $F^{(m)}(V)$ is irreducible.

Note that, if $a \in \mathfrak{gl}_\infty$ then $a \in \mathfrak{gl}_{st}$ for some $s \leqslant t$, and so for any $\xi \in \mathbb{R}$ the one parameter subgroup $\exp(\xi a) \subset GL(s,t)$, and hence, using standard techniques, [3, 14], one can show the following.

Proposition 2.2. $\tilde{L}(\exp(a)) = \exp(\tilde{\ell}(a))$ for $a \in \mathfrak{gl}_\infty$.

Corollary 2.3. If $\tilde{\ell}$ is an irreducible representation of $\mathfrak{gl}_\infty$ on a vector space X, then the representation $\tilde{L}$ of GL_∞ on X is also irreducible.

Proposition 2.4 (Kac). Define an involution on $\mathfrak{gl}_\infty$ by $a \mapsto a^\dagger$ (conjugate transpose). Then with respect to this involution, the representation $\tilde{\ell}$ on $F(V)$ is 'unitary'. That is

$$\langle \tilde{\ell}(a)(v)|w\rangle = \langle v|\tilde{\ell}(a^\dagger)(w)\rangle \qquad a \in \mathfrak{gl}_\infty \qquad v, w \in F(V)$$

2.3. Weights and weight vectors in $F(V)$

Let $\mathcal{N}_+$ be the upper triangular nilpotent subalgebra of $\mathfrak{gl}_\infty$.

$$\mathcal{N}_+ = \{(a_{ij}) \in \mathfrak{gl}_\infty | a_{ij} = 0 \text{ for } j \leqslant i\}.$$

Then, since $\tilde{\ell}(E_{ij})v_{\underline{M}} = 0$ for $i < j$ we have

$$\tilde{\ell}(\mathcal{N}_+)v_{\underline{M}} = 0$$
$$\tilde{\ell}(E_{ii})v_{\underline{M}} = \lambda_i v_{\underline{M}}$$

where

$$\lambda_i = \begin{cases} 1 & i \leqslant m \\ 0 & i > m \end{cases}$$

In addition we also have

$$\tilde{\ell}_m(E_{ii})v_I = \lambda_i v_I$$

for every basis vector v_I in F where,

$$\lambda_i = \begin{cases} 1 & i \in I \\ 0 & \text{otherwise.} \end{cases}$$

With this in mind, given a collection of numbers

$$\lambda = \{\lambda_i | i \in \mathbb{Z}\}$$

we define a *simple weight*

$$\omega_I = \{\lambda_i, i \in \mathbb{Z} | \lambda_i = 1 \text{ if } i \in I,\ 0 \text{ otherwise}\} \tag{5}$$

for any shuffle I, and we have the following definition due to [10].

Definition 2.5. Given a collection of numbers $\lambda = \{\lambda_i | i \in \mathbb{Z}\}$ called a *highest weight*, define the *irreducible highest weight representation* π_λ of $\mathfrak{gl}_\infty$ as an irreducible representation on a vector space $L(\lambda)$ which admits a non-zero vector v_λ where

$$\pi_\lambda(\mathcal{N}_+)v_\lambda = 0$$
$$\pi_\lambda(E_{ii})v_\lambda = \lambda_i v_\lambda.$$

The vector v_λ is called the *highest weight vector*.

In view of the above definition, we see that for any $m \in \mathbb{Z}$, $v_{\underline{M}}$ is a highest weight vector of the representation $\tilde{\ell}$, with highest weight

$$\omega_{\underline{M}} = \{\lambda_i | \lambda_i = 1 \text{ for } i \leqslant m,\ \lambda_i = 0 \text{ for } i > m\}.$$

Weights of this form are called *fundamental weights*, which can be visualized as

$$\omega_{\underline{M}} = (\ldots 1, 1, 1, 0, 0, 0, \ldots)$$

with the last 1 appearing in the mth position.

Proposition 2.6 (Kac). The highest weight characterizes the representation up to equivalence.

By corollary 2.3, for each $m \in \mathbb{Z}$, the representation $\tilde{L}$ of GL_∞ on $F^{(m)}(V)$ is also irreducible.

2.4. Kronecker and tensor product representations

Given the representation $\tilde{L}$ of GL_∞ on $F(V)$ we define the Kronecker product (or interior tensor product) of the representations $\tilde{L}$ (which we also call $\tilde{L}$) of GL_∞ on $F(V) \bigotimes F(V)$ by

$$\tilde{L}(A)(\psi_1 \otimes \psi_2) = \tilde{L}(A)\psi_1 \otimes \tilde{L}(A)\psi_2 \qquad \text{for any } \psi_1, \psi_2 \in F(V)$$

and the associated Lie algebra representation (which we also call $\tilde{\ell}$) of $\mathfrak{gl}_\infty$ on $F(V) \bigotimes F(V)$ by

$$\tilde{\ell}(a)(\psi_1 \otimes \psi_2) = \tilde{\ell}(a)\psi_1 \otimes \psi_2 + \psi_1 \otimes \tilde{\ell}(a)\psi_2.$$

This idea extends inductively to finite tensor products of $F(V)$.

Now if ψ_1 and ψ_2 are weight vectors of $F(V)$ with corresponding weights

$$\omega_{I_1} = \{\lambda_{1,i} | \lambda_{1,i} = 1 \text{ if } i \in I_1,\ 0 \text{ otherwise}\}$$

and

$$\omega_{I_2} = \{\lambda_{2,i} | \lambda_{2,i} = 1 \text{ if } i \in I_2,\ 0 \text{ otherwise}\}$$

then $\psi_1 \otimes \psi_2$ is also a weight vector with weight $\omega_{I_1} + \omega_{I_2}$. Since $\nu_{\underline{M}_1}$ and $\nu_{\underline{M}_2}$ are highest weight vectors with highest weights $\omega_{\underline{M}_1}$ and $\omega_{\underline{M}_2}$, then $\nu_{\underline{M}_1} \otimes \nu_{\underline{M}_2}$ is the highest weight vector with highest weight $\omega_{\underline{M}_1} + \omega_{\underline{M}_2}$.

One proves the following in a similar way to the proof of proposition 2.1.

Proposition 2.7. The action $\tilde{\ell}$ of $\mathfrak{gl}_\infty$ on $\nu_{\underline{M}_1} \otimes \nu_{\underline{M}_2}$ generates an irreducible highest weight subrepresentation of $\mathfrak{gl}_\infty$ contained in $F^{(m_1)}(V) \bigotimes F^{(m_2)}(V)$, the highest weight submodule, with highest weight $\omega_{\underline{M}_1} + \omega_{\underline{M}_2}$. By corollary 2.3 this is also an irreducible representation of GL_∞.

Given a vector space such as $F(V)$ we can form the *tensor algebra*

$$\mathcal{T}(F(V)) = \sum_{n=0}^{\infty} \oplus (F(V))^{\otimes n}$$

(where $n = 0$ corresponds to $\mathbb{C}$) and the representations $\tilde{L}$ and $\tilde{\ell}$ extend as above to representations on $\mathcal{T}(F(V))$.

We summarize the required results of [10] with the following theorem.

Theorem 2.8 (Kac). All irreducible unitary highest weight representations of $\mathfrak{gl}_\infty$, and therefore of GL_∞, in $\mathcal{T}(F(V))$ are characterized up to equivalence by highest weights of the form

$$\omega = \sum_{i=1}^{n} k_i \omega_{\underline{M}_i}$$

where the $\omega_{\underline{M}_i}$ are fundamental weights and the k_i are non-negative integers. Each such representation can be realized as the module generated by the action $\tilde{\ell}$ of $\mathfrak{gl}_\infty$ on the highest weight vector

$$\nu_\omega = \nu_{\underline{M}_1}^{\otimes k_1} \otimes \cdots \otimes \nu_{\underline{M}_n}^{\otimes k_n}.$$

We denote this highest weight module by $F^{(\omega)}(V)$ and with the obvious modifications, we obtain identical results for representations on $F(V^t)$, and obtain highest weight module $F^{(\omega)}(V^t)$.

Given the representations $\tilde{L}$ and $\tilde{R}$ of GL_∞ on $F(V)$ and $F(V^t)$ respectively, we obtain the (exterior) tensor product representation $\tilde{L} \bigotimes \tilde{R}$ of $GL_\infty \times GL_\infty$ on $F(V) \bigotimes F(V^t)$

$$\tilde{L} \otimes \tilde{R}(A, B)(v_I \otimes v_J^t) = \tilde{L}(A)v_I \otimes \tilde{R}(B)v_J^t \qquad \text{for } A, B \in GL_\infty$$

and also the associated Lie algebra action of $\mathfrak{gl}_\infty \times \mathfrak{gl}_\infty$ given by

$$\tilde{\ell} \otimes \tilde{r}(a, b)(v_I \otimes v_J^t) = \tilde{\ell}(a)v_I \otimes v_J^t + v_I \otimes \tilde{r}(b)v_J^t \qquad \text{for } a, b \in \mathfrak{gl}_\infty.$$

These representations extend in the obvious way to representations on $\mathcal{T}(F(V)) \bigotimes \mathcal{T}(F(V^t))$.

Recall that v_I is a weight vector of the representation $\tilde{\ell}$ with weight ω_I and v_J^t is a weight vector of the representation $\tilde{r}$ with weight ω_J. In what follows it will be desirable to distinguish between weights of the left and right actions, so we say that $v_I \otimes v_J^t$ has a *double weight* $(\omega_I|\omega_J)$, and one proves, as in proposition 2.1, that $v_{\underline{M}_1} \otimes v_{\underline{M}_2}^t$ is a highest weight vector of the joint action $\tilde{\ell} \otimes \tilde{r}$ with highest weight $(\omega_{\underline{M}_1}|\omega_{\underline{M}_2})$. It follows by induction that $v_{\omega_1} \otimes v_{\omega_2}^t$ is a highest weight vector of $\tilde{\ell} \otimes \tilde{r}$ with highest weight $(\omega_1|\omega_2)$. Since tensor products of irreducible representations are irreducible, the module generated by the joint action $\tilde{\ell} \otimes \tilde{r}$ on $v_{\omega_1} \otimes v_{\omega_2}^t$ is the unique ($\mathfrak{gl}_\infty \times \mathfrak{gl}_\infty$ and $GL_\infty \times GL_\infty$) irreducible highest weight submodule of $\mathcal{T}(F(V)) \bigotimes \mathcal{T}(F(V^t))$ with highest weight $(\omega_1|\omega_2)$.

In what follows the case when $\omega_1 = \omega_2$, and the algebraic direct sum of irreducible highest weight submodules

$$\Omega = \sum_{\omega} \oplus (F^{(\omega)}(V) \otimes F^{(\omega)}(V^t)). \tag{6}$$

will be of particular interest.

3. Determinants, representations, and Fock spaces

3.1. Determinants in GL_∞

Let $A \in GL_\infty$. Then from the definition we see that there is a positive integer N such that

$$a_{ij} = \delta_{ij} \qquad \text{whenever } |i| > N \text{ or } |j| > N. \tag{7}$$

Denote by I^k the first k terms of the shuffle $i_m > i_{m-1} > i_{m-2} > \cdots$ and denote by $A_{J^k}^{I^k}$ the submatrix obtained from A by taking the intersection of the columns $i_m > i_{m-1} > \cdots > i_{m-k}$ and the rows $j_m > j_{m-1} > \cdots > j_{m-k}$. Then the function

$$\Delta_{J^k}^{I^k}(A) = \det(A_{J^k}^{I^k})$$

is well defined for each k. Recall that if I and J are shuffles, then for n sufficiently large

$$i_{(m-n)} = -(m+n) = j_{(m-n)} \tag{8}$$

i.e. the indices lie in the 'tail' of the shuffles, where the shuffles are no longer 'shuffled'. Also, for $n > N$ (N as in (7)), then $A_{i_{(m-n)} j_{(m-n)}} = 1$, all other entries in the $i_{(m-n)}$th row and $j_{(m-n)}$th column are zero.

Thus, for k sufficiently large, as in (8), and for $k > N$ as in (7) we have, for each $A \in GL_\infty$

$$\Delta_{J^k}^{I^k}(A) = \Delta_{J^{k+1}}^{I^{k+1}}(A) = \Delta_{J^{k+2}}^{I^{k+2}}(A) = \cdots \tag{9}$$

which leads to the following.

Definition 3.1. For any shuffles I and J, and for each $A \in GL_\infty$

$$\Delta^I_J(A) = \lim_{k\to\infty} \Delta^{I^k}_{J^k}(A).$$

In accordance with the finite-dimensional case, for each $m \in \mathbb{Z}$ we can define the *principal minors* of A as $\Delta^{\underline{M}}_{\underline{M}}(A)$. We can also define the *determinant* of A as

$$\det(A) = \lim_{m\longrightarrow\infty} \Delta^{\underline{M}}_{\underline{M}}(A).$$

It is most useful to consider the function Δ^I_J as the limit of a sequence of functions that converge pointwise on GL_∞. We also remark that I and J need not be shuffles of the same $\underline{M}$, but usually are.

The following facts are routinely verified.

Lemma 3.2. Let $A \in GL_\infty$. I, J shuffles of $\underline{M}$ for some m. Then

(1) $A\nu_i$ is the vector with entries consisting of the ith column of A, i.e.

$$A\nu_i = \sum_{k=-\infty}^{\infty} A_{ki}\nu_k$$

and all but finitely many of the $A_{ki} = 0$.

(2) If $|i| > N$, N as in (7), then A acts trivially on ν_i, i.e.

$$A\nu_i = \sum_k A_{ki}\nu_k = \sum_k \delta_{ki}\nu_k = \nu_i.$$

This corresponds with the previous notion of GL_∞ as an inductive limit.

(3) If $i_{m-k} < -N$ (N as in (7)), then $\Delta^{I_k}_{J_k}(A) = 0$ unless $i_{m-k} = j_{m-k}$.

(4) If $j_m > N$ then $\Delta^I_J = 0$ unless $i_m = j_m$.

(5) If $j_m < -N$ then $\Delta^I_J(A) = \delta^I_J$.

(6) If d is a diagonal matrix in GL_∞, then $\Delta^I_J(d) = 0$ unless $I = J$.

(7) If K is a shuffle of $\underline{M}$, and B is an upper triangular matrix with entries $\ldots, b_k, \ldots, b_{k+l}, \ldots$ on the principal diagonal, then $\Delta^{\underline{M}}_K(B) = 0$ unless $K = \underline{M}$, in which case $\Delta^{\underline{M}}_{\underline{M}}(B) = \prod_{j\in\underline{M}} b_j$. It follows that if K is a shuffle of $\underline{M}$, and ζ is an upper triangular matrix with ones on the diagonal, then $\Delta^{\underline{M}}_K(\zeta) = 1$ if $K = \underline{M}$ and zero otherwise, since clearly $\Delta^{\underline{M}}_{\underline{M}}(\zeta) = 1$. Similarly for the transpose $\Delta^K_{\underline{M}}(\zeta^t)$.

The following appears in [10] without proof or further comment. A proof is included here for completeness.

Lemma 3.3. If, as in (3), we define a left action of GL_∞ on $F(V)$ by

$$\tilde{L}(A)(\nu_I) = A\nu_{i_m} \wedge A\nu_{i_{m-1}} \wedge A\nu_{i_{m-2}} \wedge \ldots$$

then

$$\tilde{L}(A)(\nu_I) = \sum_J \Delta^I_J(A)\nu_J$$

where $J = j_m > j_{m-1} > j_{m-2} > \cdots$ and the sum is taken over all such J's. Moreover, this sum is finite.

Proof. By lemma 3.2(2) we have

$$\tilde{L}(A)\nu_I = A\nu_{i_m} \wedge A\nu_{i_{m-1}} \wedge \ldots \wedge A\nu_{i_{m-k}} \wedge \nu_{i_{m-k-1}} \wedge \ldots$$

for some k where $i_{m-k} < -N$. Without loss of generality we may assume that k is large enough so that $i_{m-n} = m - n$ for all $n > k$, i.e. i_{m-n} lies in the tail of I.

So

$$\tilde{L}(A)v_I = Av_{i_m} \wedge Av_{i_{m-1}} \ldots \wedge Av_{i_{m-k}} \wedge v_{m-(k+1)} \wedge v_{m-(k+2)} \wedge \ldots .$$

Next consider the finite wedge product

$$Av_{i_m} \wedge Av_{i_{m-1}} \wedge \ldots \wedge Av_{i_{m-k}}. \tag{10}$$

By lemma 3.2(1) each is a vector of finite length. In fact, if we set Max $= \max\{i_m, N\}$ then each vector in (10) can be written as

$$Av_i = \sum_{s=-N}^{\mathrm{Max}} A_{si} v_s$$

and so we may cite the results for finite wedge products [15] and write

$$Av_{i_m} \wedge \ldots \wedge Av_{i_{m-k}} = \sum_{\mathrm{Max} \geqslant j_m > j_{m-1} > \cdots > j_{m-k} \geqslant -N} \Delta^{i_m, i_{m-1}, \ldots, i_{m-k}}_{j_m, j_{m-1}, \ldots, j_{m-k}}(A) v_{j_m} \wedge v_{j_{m-1}} \wedge \ldots \wedge v_{j_{m-k}}$$

where the sum is taken over all such shuffles between Max and $-N$ (note that $k \leqslant [\mathrm{Max} - (-N)]$).

Thus we can write

$$\begin{aligned}
\tilde{L}(A)v_I &= \tilde{L}(A)(v_{i_m} \wedge v_{i_{m-1}} \wedge v_{i_{m-2}} \wedge \ldots) \\
&= Av_{i_m} \wedge Av_{i_{m-1}} \ldots \wedge Av_{i_{m-k}} \wedge v_{m-(k+1)} \wedge v_{m-(k+2)} \wedge \ldots \\
&= [\tilde{L}(A)v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots \wedge v_{i_{m-k}}] \wedge v_{i_{m-(k+1)}} \ldots \\
&= \left\{ \sum_{\mathrm{Max} \geqslant j_m > \cdots > j_{m-k} \geqslant -N} \Delta^{i_m, \ldots, i_{m-k}}_{j_m, \ldots, j_{m-k}}(A) v_{j_m} \wedge \ldots \wedge v_{j_{m-k}} \right\} \wedge v_{m-(k+1)} \wedge \ldots .
\end{aligned}$$

Next, as an inductive step, consider

$$\begin{aligned}
&\{Av_{i_m} \wedge \ldots \wedge Av_{i_{m-k}}\} \wedge v_{m-(k+1)} \\
&\quad = \left\{ \sum_{\mathrm{Max} \geqslant j_m > \cdots > j_{m-k} \geqslant -N} \Delta^{i_m, \ldots, i_{m-k}}_{j_m, \ldots, j_{m-k}}(A) v_{j_m} \wedge \ldots \wedge v_{j_{m-k}} \right\} \wedge v_{m-(k+1)}.
\end{aligned} \tag{11}$$

Since $m - k < -N$, by (9), we have either

$$\Delta^{i_m, i_{m-1}, \ldots, i_{m-k}}_{j_m, j_{m-1}, \ldots, j_{m-k}}(A) = \Delta^{i_m, i_{m-1}, \ldots, i_{m-k}, i_{m-(k+1)}}_{j_m, j_{m-1}, \ldots, j_{m-k}, j_{m-(k+1)}}(A)$$

or, if we also have $j_{m-(k+1)} \neq m - (k+1)$ $(= i_{m-(k+1)})$ then by lemma 3.2(3)

$$\Delta^{i_m, i_{m-1}, \ldots, i_{m-k}, i_{m-(k+1)}}_{j_m, j_{m-1}, \ldots, j_{m-k}, j_{m-(k+1)}}(A) = 0$$

and so in either case we can write (11) as

$$\sum_{\mathrm{Max} \geqslant j_m > j_{m-1} > \cdots > j_{m-k} > j_{m-(k+1)} \geqslant -(N+1)} \Delta^{i_m, i_{m-1}, \ldots, i_{m-k}, i_{m-(k+1)}}_{j_m, j_{m-1}, \ldots, j_{m-k}, j_{m-(k+1)}}(A) v_{j_m} \wedge v_{j_{m-1}} \wedge \ldots \wedge v_{j_{m-k}} \wedge v_{m-(k+1)}.$$

Thus for n, sufficiently large,

$$\begin{aligned}
&\tilde{L}(A)(v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots \wedge v_{i_{m-n}} \wedge v_{i_{m-(n+1)}} \wedge \ldots) \\
&\quad = \left\{ \sum_{\mathrm{Max} \geqslant j_m > j_{m-1} > \cdots > j_{m-n}} \Delta^{i_m, i_{m-1}, \ldots, i_{m-n}}_{j_m, j_{m-1}, \ldots, j_{m-n}}(A) \right. \\
&\qquad \left. \times (v_{j_m} \wedge v_{j_{m-1}} \wedge \ldots \wedge v_{j_{m-n}}) \right\} \wedge v_{j_{m-(n+1)}} \wedge \ldots \\
&\quad = \left\{ \sum_{\mathrm{Max} \geqslant j_m > j_{m-1} > \cdots > j_{m-(n+1)}} \Delta^{i_m, i_{m-1}, \ldots, i_{m-n}, i_{m-(n+1)}}_{j_m, j_{m-1}, \ldots, j_{m-n}, j_{m-(n+1)}}(A) \right. \\
&\qquad \left. \times (v_{j_m} \wedge v_{j_{m-1}} \wedge \ldots \wedge v_{j_{m-n}} \wedge v_{j_{m-(n+1)}}) \right\} \wedge v_{j_{m-(n+2)}} \wedge v_{j_{m-(n+3)}} \wedge \ldots .
\end{aligned}$$

By letting $n \to \infty$ we have

$$\tilde{L}(A)(\nu_{i_m} \wedge \nu_{i_{m-1}} \wedge \nu_{i_{m-2}} \wedge \ldots) = \sum_{\mathrm{Max} \geqslant j_m > j_{m-1} > \cdots} \Delta^{i_m, i_{m-1}, \cdots}_{j_m, j_{m-1}, \ldots}(A)(\nu_{j_m} \wedge \nu_{j_{m-1}} \wedge \ldots). \tag{12}$$

Now, since J is a shuffle of $\underline{M}$, j_m has to be greater than or equal to m for any J. Also, whenever $j_m \geqslant$ Max, then $\Delta^I_J(A) = 0$, so this is a finite sum, and we can take this sum over all $J = j_m > j_{m-1} > j_{m-2} > \cdots$. In the sequel, it will be convenient to not have to keep track of the J's in this sum, and we can write (12) more concisely as:

$$\tilde{L}(A)\nu_I = \sum_J \Delta^I_J(A)\nu_J.$$

For future reference we record that

$$\tilde{R}(A)\nu^t_I = \sum_J \Delta^I_J(A)\nu^t_J.$$

□

From the inductive limit topology we clearly have the following.

Proposition 3.4. For I, J shuffles of $\underline{M}$, the function $\Delta^I_J : GL_\infty \longrightarrow \mathbb{C}$ is continuous.

By [20] the representation $\tilde{L}$ is continuous if $\Longleftrightarrow$ the map

$$\phi_\nu : GL_\infty \longrightarrow F(V)$$

given by

$$A \mapsto \tilde{L}(A)\nu$$

is continuous for each $\nu \in F(V)$. Checking on the basis elements we have

$$\phi_{\nu_I}(A) = \tilde{L}(A)\nu_I = \sum_J \Delta^I_J(A)\nu_J$$

and so we have the following.

Corollary 3.5. The action $\tilde{L}$ of GL_∞ on $F^{(m)}(V)$ is continuous. Thus $\tilde{L}$ and similarly $\tilde{R}$ are representations of the topological group GL_∞ on $F(V)$ and $F(V^t)$ respectively.

The following lemma will be used often in the sequel.

Lemma 3.6. For shuffles I, J of $\underline{M}$ for some m, $A, B \in GL_\infty$

$$\Delta^I_K(BA) = \sum_J \Delta^I_J(A)\Delta^J_K(B).$$

Proof. Let

$$\nu_I = \nu_{i_m} \wedge \nu_{i_{m-1}} \wedge \nu_{i_{m-2}} \wedge \ldots.$$

Then since $\tilde{L}$ is an action on $F(V)$ we have

$$\tilde{L}(BA)(\nu_I) = \tilde{L}(B)\tilde{L}(A)(\nu_I). \tag{13}$$

Applying lemma 3.3 to both sides of (13) we have

$$\sum_K (\Delta^I_K(BA))\nu_K = \sum_K \left(\sum_J \Delta^I_J(A)\Delta^J_K(B)\right)\nu_K.$$

Since the ν_K's form a basis of $F(V)$ and all the sums in sight are finite, we may equate the coefficients in parenthesis to obtain

$$\Delta^I_K(BA) = \sum_J \Delta^I_J(A)\Delta^J_K(B).$$

□

3.2. Projective and inductive limits of Fock spaces

Consistent with the definition of a continuous function, a function $f : GL_\infty \longrightarrow \mathbb{C}$ is *analytic* in the inductive limit topology if the restriction

$$f|_{GL(s,t)} :\longrightarrow \mathbb{C}$$

is analytic for all $s \leqslant t$. For reasons to be apparent, we define the set we call G *augmented* by

$$G^{\text{aug}} = \{A = (A_{ij})_{i,j\in\mathcal{Z}} | A_{ij} - \delta_{ij} = 0 \text{ for all but finitely many } i, j, \\ \text{but } A \text{ need not be invertible}\}$$

which we take to be the inductive limit of the subsets

$$G^{\text{aug}}_{st} = \{A \in G^{\text{aug}} | A_{ij} - \delta_{ij} = 0, \ \text{unless } s \leqslant i, \ \ j \leqslant t\}.$$

Since each $GL(s,t)$ is an open, dense subset of G^{aug}_{st}, GL_∞ is an open, dense subset of G^{aug}, and clearly we may identify G^{aug}_{st} with $\mathbb{C}^{(t-s)^2}$.

We define a Gaussian measure on $\mathbb{C}^{(t-s)^2}$ as in [11] by setting

$$\mathrm{d}\mu = \pi^{-(t-s)^2} \exp(-\text{trace}\, ZZ^\dagger) dZ$$

where dZ is the Lebesgue product measure on $\mathbb{C}^{(t-s)^2}$. We then say a function $f : G^{\text{aug}} \longrightarrow \mathbb{C}$ is *square integrable on* G^{aug} if

$$\int_{G^{\text{aug}}_{st}} |f(Z)|^2 \,\mathrm{d}\mu(Z) < \infty \qquad \text{for all } s \leqslant t$$

and denote by $\mathcal{F}$ the set of analytic square-integrable functions on G^{aug}. Given a function $f \in \mathcal{F}$, we will denote by f_{st} the restriction of f to G^{aug}_{st} and similarly denote by $\mathcal{F}_{s,t}$ the restriction of $\mathcal{F}$ to G^{aug}_{st}. Thus, a function $f : G^{\text{aug}} \longrightarrow \mathbb{C}$ is analytic square integrable if and only if f_{st} is analytic square integrable for all $s \leqslant t$. Certainly $\mathcal{F}_{s,t} \subset \mathcal{F}$ for all $s \leqslant t$, and each $\mathcal{F}_{s,t}$ is a Hilbert space with respect to the inner product

$$\langle f|g\rangle_{s,t} = \int_{G^{\text{aug}}_{st}} f(Z)\overline{g(Z)}\,\mathrm{d}\mu(Z) \tag{14}$$

and the norm

$$\|f\|_{s,t} = \int_{G^{\text{aug}}_{st}} |f(Z)|^2 \,\mathrm{d}\mu(Z).$$

As before, the collection $\{(s,t)\}_{s\leqslant t}$ is a directed set, and along with the restriction maps

$$\text{Res}_{st} : \mathcal{F} \longrightarrow \mathcal{F}_{s,t}$$
$$\text{Res}^{s't'}_{st} : \mathcal{F}_{s',t'} \longrightarrow \mathcal{F}_{s,t}$$

we have that $\mathcal{F}$ is the *inverse limit* of the Hilbert spaces $\mathcal{F}_{s,t}$.

$$\mathcal{F} = \varprojlim \mathcal{F}_{s,t}.$$

We then give $\mathcal{F}$ the topology of the *projective limit*†, the topology for which the sets

$$\{(\text{Res}_{st})^{-1}(U_{st}) | U_{st} \text{ open in } \mathcal{F}_{s,t}\}$$

form a base, and write

$$\mathcal{F} = \underset{\longleftarrow}{\text{Proj}}\, \mathcal{F}_{s,t}.$$

As a direct result we have the following.

† Again, we use different terminology to distinguish between an object's algebraic and topological incarnations.

Proposition 3.7. The restriction maps $\mathrm{Res}_{st} : \mathcal{F} \longrightarrow \mathcal{F}_{s,t}$ are continuous, and a sequence f_n converges to a function f in $\mathcal{F}$ if and only if $(f_n)_{st}$ converges to f_{st} for all $s \leqslant t$.

Consistent with the definition of continuous and analytic functions, we also define the space

$$\mathcal{P} = \{p : G^{\mathrm{aug}} \longrightarrow \mathbb{C} | p|_{G^{\mathrm{aug}}_{st}} \text{ is a polynomial for all } s \leqslant t\}.$$

Note that the functions Δ^I_J are in $\mathcal{P}$, and certainly $\mathcal{P} \subset \mathcal{F}$. We set

$$\mathcal{P}_{s,t} = \mathcal{P}|_{G^{\mathrm{aug}}_{st}}$$

and since each $\mathcal{P}_{s,t}$ is a dense, open subset of $\mathcal{F}_{s,t}$, $\mathcal{P}$ is a dense, open subset of $\mathcal{F}$. Furthermore, each $\mathcal{P}_{s,t}$ inherits the inner product from $\mathcal{F}_{s,t}$ and it is well known (cf [11]) that the inner product defined by (14), when restricted to $\mathcal{P}$, is equivalent to that defined by

$$(p_{st}|q_{st}) = p_{st}(D)\overline{q_{st}(\overline{Z})}|_{Z=0} \tag{15}$$

where $q(D)$ denotes the differential operator obtained by formally replacing Z_{ij} by the partial derivative $\partial/\partial Z_{ij}$. This inner product is often easier to compute.

Given that $\mathcal{F}$ is the projective limit of Hilbert spaces, we wish to extend the notion of orthogonality in a meaningful and useful way. Note that even though $f \neq g \in \mathcal{F}$, we can have $f_{st} = g_{st}$ for some $s \leqslant t$, with $t - s$ small enough. This motivates the following definition.

Definition 3.8. $f, g \in \mathcal{F}$ are orthogonal if $\langle f|g\rangle_{s,t} = 0$ for all but finitely many $s \leqslant t$†.

The following lemma illustrates this idea.

Lemma 3.9. For any $m \in \mathbb{Z}$ let I and I' (or J and J') be shuffles of $\underline{M}$ with $I \neq I'$ (respectively $J \neq J'$). Then the functions Δ^I_J and $\Delta^{I'}_J$ (respectively Δ^I_J and $\Delta^I_{J'}$) are orthogonal.

Proof. If I is a shuffle of $\underline{M}$ then, as in section 3, denote by I^k the first k terms of I

$$I^k = i_m > i_{m-1} > \cdots > i_{m-k}.$$

We first show that if I^k and I'^k are as above, with $I^k \neq I'^k$, and if $\langle\ |\ \rangle$ is the inner product given by (15), then $\langle \Delta^{I^k}_{J^k} | \Delta^{I'^k}_{J^k}\rangle = 0$. (Similarly, if $J^k \neq J'^k$ then $\langle \Delta^{I^k}_{J^k} | \Delta^{I^k}_{J'^k}\rangle = 0$.)

The proof is by induction on k. If $k = 1$, with $I^k \neq I'^k$ then

$$\Delta^{I^k}_{J^k} = \Delta^{i_m}_{j_m} = z_{i_m j_m} \neq z_{i'_m j_m} = \Delta^{i'_m}_{j_m} = \Delta^{I'^k}_{J^k}$$

and

$$\langle \Delta^{i_m}_{j_m} | \Delta^{i'_m}_{j_m}\rangle = \partial/\partial z_{i_m j_m}(z_{i'_m j_m}) = 0.$$

Now suppose the hypothesis is true for all determinants with $k-1$ rows and columns, and expand $\Delta^{I^k}_{J^k}$ by minors. The result then follows directly from the induction hypothesis.

It is then clear that if I and I' are shuffles with $I \neq I'$ (respectively $J \neq J'$), and if $\langle\ |\ \rangle_{s,t}$ is the inner product (14) restricted to $\mathcal{P}_{s,t}$, then $\langle \Delta^I_J | \Delta^{I'}_J\rangle_{s,t} = 0$ (respectively $\langle \Delta^I_J | \Delta^I_{J'}\rangle_{s,t} = 0$) for all s sufficiently small and all t sufficiently large. Hence the two functions are orthogonal. □

† We do not attempt to extend this definition to obtain an inner product or norm on $\mathcal{F}$, and for any given $f \in \mathcal{F}$, $\sup_{s \leqslant t} \|f\|_{s,t}$ need not be bounded. The interested reader is invited to compare this situation with that in [19].

Next we note that for $s' \leqslant s \leqslant t \leqslant t'$, we have the inclusion maps

$$\mathrm{Inc}_{st}^{s't'} : \mathcal{F}_{s,t} \longrightarrow \mathcal{F}_{s',t'}$$

and so can take the *direct limit* of the Hilbert spaces $\mathcal{F}_{s,t}$

$$\mathfrak{F}_0 = \varinjlim \bigcup_{s \leqslant t} \mathcal{F}_{s,t}$$

and denote the inclusion map by $\mathrm{Inc}_{st} : \mathcal{F}_{s,t} \longrightarrow \mathfrak{F}_0$. Since

$$\langle\ |\ \rangle_{s',t'}|_{\mathcal{F}_{s,t}} = \langle\ |\ \rangle_{s,t}$$

the inclusions are isometric embedding of $\mathcal{F}_{s,t}$ into $\mathcal{F}_{s',t'}$, so there is no loss of generality when denoting by $\|\cdot\|$ and $\langle\ |\ \rangle$ the norm and the inner product in every $\mathcal{F}_{s,t}$ and also in $\mathfrak{F}_0$. We then set

$$\mathfrak{F} = \overline{\varinjlim \bigcup_{s \leqslant t} \mathcal{F}_{s,t}}$$

where the bar indicates the Hilbert space completion with respect to the norm, so that $\mathfrak{F}$ is a Hilbert space, and say $\mathfrak{F}$ is the *inductive limit*, denoted by

$$\mathfrak{F} = \underset{\longrightarrow}{\mathrm{Ind}}\, \mathcal{F}_{s,t}$$

of the Hilbert spaces $\mathcal{F}_{s,t}$. We remark that the isometric embeddings makes the inclusion maps continuous. Analogously we set

$$\mathfrak{P}_0 = \varinjlim \mathcal{P}_{s,t}$$

and take $\mathfrak{P}$ to be the topological subspace of $\mathfrak{F}$. Again $\mathfrak{P}$ is an open, dense subset of $\mathfrak{F}$, since $\mathcal{P}_{s,t} = (\mathrm{Inc}_{st})^{-1}(\mathfrak{P})$ is open and dense in $\mathcal{F}_{s,t}$ for all $s \leqslant t$.

4. Representations of GL_∞ in $\mathcal{F}$

4.1. Regular representations

Definition 4.1. Let $\mathcal{F}$ be as in section 3.2, and let $G = GL_\infty$. For $f, g \in \mathcal{F}$, $A, B \in G$ we define the *right regular representation* of G on $\mathcal{F}$ by

$$R(A)f = f_A$$

where $f_A(z) := f(zA)$ for all $z \in G^{\mathrm{aug}}$. Similarly we define the *left regular representation* by

$$L(B)f = f_{B^t}$$

where $f_{B^t}(z) := f(B^t z)$ for all $z \in G^{\mathrm{aug}}$. It is routine to verify that these are continuous actions of G on $\mathcal{F}$ that commute. We give $\mathcal{F}$ a bimodule structure via the joint action of $G \times G$ on $\mathcal{F}$ by

$$L \odot R(A, B)f(z) = [L(A) \odot R(B)]f(z) = f_{A^t B}(z)$$

where $f_{A^t B}(z) := f(A^t z B)$.

Our aim is to decompose $\mathcal{F}$ under this action.

Remark 4.2. We can also define the (exterior) tensor product of representations on the space $\mathcal{F} \bigotimes \mathcal{F}$ by

$$L \otimes R(A, B)[f \otimes g](z) = L(A)f(z) \otimes R(B)g(z).$$

4.2. Weights and weight vectors in $\mathcal{F}$

Let D denote the diagonal subgroup of $G = GL_\infty$. That is

$$D = \{(d_{i,j}) \in G | d_{i,j} = 0 \text{ for } i \neq j\}$$

and for $l \in \mathbb{Z}$ let $\alpha = (\ldots, \alpha_l, \alpha_{l+1}, \alpha_{l+2}, \ldots)$ be a sequence of non-negative integers. For each α define a holomorphic character $\pi^{(\alpha)} : D \longrightarrow \mathbb{C}^*$ (the non-zero complex numbers) by

$$\pi^{(\alpha)}(d) = \prod_{i \in \mathbb{Z}} d_i^{\alpha_i} \qquad \text{for all } d \in D.$$

Note that $\pi^{(\alpha)}(d)$ is well defined for each $d \in D$.

We say f is a *weight vector* for the representation L on $\mathcal{F}$ with weight α if

$$L(d) f(z) = \pi^{(\alpha)}(d) f(z) \qquad \text{for all } d \in D$$

and we sometimes say that f *transforms* L*-covariantly* with respect to $\pi^{(\alpha)}$. We say that f is a *highest weight vector* for the representation L if f is a weight vector and if

$$L(\zeta) f(z) = f(z) \qquad \text{for all } \zeta \in Z_+$$

where Z_+ is the subgroup of G consisting of upper triangular matrices with ones along the principal diagonal.

It is routine to verify that the space of functions that transform L-covariantly with respect to $\pi^{(\alpha)}$ form a subspace of $\mathcal{F}$ which is invariant under the right action, R, of G on $\mathcal{F}$. With the obvious modifications we define weight vectors and highest weight vectors for the right action R of G on $\mathcal{F}$, obtaining a subspace of R-covariant vectors that is invariant under the left action L. Since we wish to distinguish between the weights of the left and right actions, we say that $f \in \mathcal{F}$ is a weight vector of the joint action $L \odot R$ with weight $(\alpha_1 | \alpha_2)$ if

$$L \odot R(d_1, d_2) f(z) = \pi^{(\alpha_1)}(d_1) \pi^{(\alpha_2)}(d_2) f(z).$$

Similarly we say that $f_1 \otimes f_2 \in \mathcal{F} \otimes \mathcal{F}$ is a weight vector of the tensor product action $L \otimes R$ with weight $(\alpha_1 | \alpha_2)$ if

$$L \otimes R(d_1, d_2) f_1 \otimes f_2(z) = \pi^{(\alpha_1)}(d_1) \pi^{(\alpha_2)}(d_2) f_1 \otimes f_2(z).$$

Proposition 4.3. For each $m \in \mathbb{Z}$, and any I, J shuffles of $\underline{M}$, the function Δ_J^I is a weight vector of the representation L with weight ω_J and a weight vector of the representation R with weight ω_I, where ω_I and ω_J are the simple weights defined in (5). Thus, Δ_J^I is also a weight vector for the joint action $L \odot R$ with weight $(\omega_I | \omega_J)$.

Proof. We prove the case for R.

Let $d \in D$, $z \in G^{\text{aug}}$, then

$$\begin{aligned}
& L(d)\Delta_J^I(z) = \Delta_J^I(d^t z) \\
\text{since } d^t = d \qquad & = \Delta_J^I(dz) \\
\text{by lemma 3.6} \qquad & = \sum_K \Delta_J^K(d) \Delta_K^I(z) \\
\text{since } \Delta_J^K(d) = 0 \text{ unless } K = J \qquad & = \Delta_J^J(d) \Delta_J^I(z) \\
& = \pi^{(\omega_J)}(d) \Delta_J^I(z).
\end{aligned}$$

□

Corollary 4.4. For each $m \in \mathbb{Z}$, the function $\Delta^{\underline{M}}_{\underline{M}}$ is a highest weight vector for both the representations L and R, with highest weight $\omega_{\underline{M}}$, where $\omega_{\underline{M}}$ is the fundamental weight defined in section 2.3. Thus, $\Delta^{\underline{M}}_{\underline{M}}$ is also a highest weight vector for the representation $L \bigodot R$ with highest weight $(\omega_{\underline{M}}|\omega_{\underline{M}})$. In addition, $\Delta^{\underline{M}}_{\underline{M}} \otimes \Delta^{\underline{M}}_{\underline{M}}$ is a highest weight vector of the representation $L \otimes R$ with highest weight $(\omega_{\underline{M}}|\omega_{\underline{M}})$.

Using a similar argument we have the following.

Proposition 4.5. For $m_1, m_2 \in \mathbb{Z}$ and for I_1, J_1 shuffles of $\underline{M}_1$ and I_2, J_2 shuffles of $\underline{M}_2$, the function $\Delta^{I_1}_{J_1}\Delta^{I_2}_{J_2}$ is a weight vector of L with weight $\omega_{I_1} + \omega_{I_2}$, a weight vector of R with weight $\omega_{J_1} + \omega_{J_2}$, and so a weight vector of the joint action $L \bigodot R$ with weight $(\omega_{I_1} + \omega_{I_2}|\omega_{J_1} + \omega_{J_2})$.

Corollary 4.6. For any $m_1, m_2 \in \mathbb{Z}$, the function $\Delta^{\underline{M}_1}_{\underline{M}_1}\Delta^{\underline{M}_2}_{\underline{M}_2}$ is a highest weight vector of the representations L and R, with weight $\omega_{\underline{M}_1} + \omega_{\underline{M}_2}$, and so also a highest weight vector of the joint action $L \bigodot R$ with weight $(\omega_{\underline{M}_1} + \omega_{\underline{M}_2}|\omega_{\underline{M}_1} + \omega_{\underline{M}_2})$. By induction, the function

$$\Delta^{(\omega)} = \left(\Delta^{\underline{M}_1}_{\underline{M}_1}\right)^{k_1} \cdots \left(\Delta^{\underline{M}_n}_{\underline{M}_n}\right)^{k_n}$$

is a highest weight vector of the representations L and R with highest weight

$$\omega = \sum_{i=1}^{n} k_i \omega_{\underline{M}_i}$$

and a highest weight vector of the joint action $L \bigodot R$ with weight $(\omega|\omega)$.

The following result is fundamental.

Theorem 4.7. The map $\Phi^{(m)} : F^{(m)}(V) \longrightarrow \mathcal{F}$ defined on basis elements by $v_I \longmapsto \Delta^{\underline{M}}_I$ (and extending by linearity) intertwines the representations $\tilde{L}$ and L. Similarly, the map $\Psi^{(m)} : F^{(m)}(V^t) \longrightarrow \mathcal{F}$ defined on basis elements by $v^t_J \longmapsto \Delta^J_{\underline{M}}$ (and extending by linearity) intertwines the representations $\tilde{R}$ and R.

Proof. We prove for the map $\Phi^{(m)}$. Let $A \in G$, $v_I \in F^{(m)}(V)$. Then for any $z \in G^{\mathrm{aug}}$ we have

$$\begin{aligned}
&\Phi^{(m)}[\tilde{L}(A)v_I](z) \\
\text{by lemma 3.3} \quad &= \Phi^{(m)}[\sum_J \Delta^I_J(A)v_J](z) \\
\text{by linearity} \quad &= \sum_J \Delta^I_J(A)\Phi^{(m)}[v_J](z) \\
&= \sum_J \Delta^I_J(A)\Delta^{\underline{M}}_J(z) \\
&= \sum_J \Delta^J_I(A^t)\Delta^{\underline{M}}_J(z) \\
\text{by lemma 3.6} \quad &= \Delta^{\underline{M}}_I(A^t z) \\
&= L(A)\Delta^{\underline{M}}_I(z) \\
&= L(A)\Phi^{(m)}[v_I](z).
\end{aligned}$$

□

Remark 4.8. The above theorem is true if, for example, we use the map $v_I \longmapsto \Delta^K_I$ for any fixed shuffle K of $\underline{M}$. We will see that the choice of $\underline{M}$ instead of K will give us the 'lowest highest weight copies' of $F^{(m)}(V)$ and $F^{(m)}(V^t)$ in $\mathcal{F}$.

Since for each $m \in \mathbb{Z}$, the spaces $F^{(m)}(V)$ and $F^{(m)}(V^t)$ are irreducible representations of G, we have the following.

Corollary 4.9. For each $m \in \mathbb{Z}$, let $\mathcal{L}_{\omega_{\underline{M}}}$ be the image of $\Phi^{(m)}(F^{(m)}(V))$. Then $\mathcal{L}_{\omega_{\underline{M}}}$ is the left submodule of $\mathcal{F}$ generated by the action L of G on the highest weight vector $\Delta^{\underline{M}}_{\underline{M}}$, with highest weight $\omega_{\underline{M}}$, and $\mathcal{L}_{\omega_{\underline{M}}}$ is an irreducible highest weight representation of G in $\overline{\mathcal{F}}$. Similarly, if $\mathcal{R}_{\omega_{\underline{M}}}$ is the image of $\Psi^{(m)}(F^{(m)}(V^t))$, then $\mathcal{R}_{\omega_{\underline{M}}}$ is the right submodule of $\mathcal{F}$ generated by the action R of G on the highest weight vector $\Delta^{\underline{M}}_{\underline{M}}$ also with highest weight $\omega_{\underline{M}}$, and $\mathcal{R}_{\omega_{\underline{M}}}$ is an irreducible highest weight representation of $\overline{G}$ in $\mathcal{F}$.

By a routine application of lemmas 3.6 and 3.9 we have the following.

Proposition 4.10. If $I \neq I'$ (respectively $J \neq J'$), then all left (respectively right) translations of Δ^I_J are $\underleftarrow{\text{orthogonal}}$ to all left (respectively right) translations of $\Delta^{I'}_J$ (respectively $\Delta^I_{J'}$). Hence $\mathcal{L}_{\omega_{\underline{M}_1}}$ and $\mathcal{L}_{\omega_{\underline{M}_2}}$ (respectively $\mathcal{R}_{\omega_{\underline{M}_1}}$ and $\mathcal{R}_{\omega_{\underline{M}_2}}$) are $\underleftarrow{\text{orthogonal}}$ whenever $\underline{M}_1 \neq \underline{M}_2$.

The next several propositions expand on these ideas.

Proposition 4.11. Let I_1 be a shuffle of $\underline{M}_1$ and I_2 be a shuffle of $\underline{M}_2$. Then the map

$$\Phi^{(m_1)} \otimes \Phi^{(m_2)} : F^{(m_1)}(V) \otimes F^{(m_2)}(V) \longrightarrow \mathcal{F}$$

given by

$$\nu_{I_1} \otimes \nu_{I_2} \longmapsto \Delta^{\underline{M}_1}_{I_1} \Delta^{\underline{M}_2}_{I_2}$$

intertwines the representation $\tilde{L}$ on $F(V) \bigotimes F(V)$ and the representation L on $\mathcal{F}$.

By induction, the map $\Phi^{(m_1)} \otimes \cdots \otimes \Phi^{(m_n)} : F^{(m_1)}(V) \otimes \cdots \otimes F^{(m_n)}(V) \longrightarrow \mathcal{F}$ defined on basis elements by

$$\nu_{I_1} \otimes \cdots \otimes \nu_{I_n} \longmapsto \Delta^{\underline{M}_1}_{I_1} \ldots \Delta^{\underline{M}_n}_{I_n}$$

intertwines $\tilde{L}$ and L.

Furthermore, for each $m \in \mathbb{Z}$ the map

$$\Phi^{(m)} \odot \Psi^{(m)} : F^{(m)}(V) \otimes F^{(m)}(V^t) \longrightarrow \mathcal{F}$$

defined on basis elements by

$$\nu_I \otimes \nu^t_J \longmapsto \Delta^J_I$$

intertwines the representation $\tilde{L} \otimes \tilde{R}$ of GL_∞ on $F^{(m)}(V) \otimes F^{(m)}(V^t)$ and the representation $L \odot R$ of GL_∞ on $\mathcal{F}$.

Proof. Apply lemmas 3.3, 3.6 and the appropriate definitions. $\square$

By iteration we have the following.

Proposition 4.12. For simplicity of notation let

$$\omega = \sum_{i=1}^{n} \oplus \omega_{\underline{M}_i}$$

where some of the $\underline{M}_i$ may be repeated. Let I_1, J_1 shuffles of $\underline{M}_1, \ldots, I_n, J_n$ shuffles of $\underline{M}_n$, and set

$$\Phi^{(\omega)} = \Phi^{(m_1)} \otimes \cdots \otimes \Phi^{(m_n)}$$
$$\Psi^{(\omega)} = \Psi^{(m_1)} \otimes \cdots \otimes \Psi^{(m_n)}.$$

Then the map

$$\Phi^{(\omega)} \odot \Psi^{(\omega)} : F^{(m_1)}(V) \otimes \cdots \otimes F^{(m_n)}(V) \bigotimes F^{(m_1)}(V^t) \otimes \cdots \otimes F^{(m_n)}(V^t) \longrightarrow \mathcal{F}$$

defined by

$$\nu_{I_1} \otimes \cdots \otimes \nu_{I_n} \bigotimes \nu^t_{J_1} \otimes \cdots \otimes \nu^t_{J_n} \longmapsto \Delta^{J_1}_{I_1} \cdots \Delta^{J_n}_{I_n}$$

intertwines $\tilde{L} \bigotimes \tilde{R}$ and $L \bigodot R$.

The following generalizes corollary 4.9.

Corollary 4.13. Let ω and $\Delta^{(\omega)}$ be as in corollary 4.6, and let $\mathcal{L}_\omega$ be the submodule of $\mathcal{F}$ generated by the left action L of G on $\Delta^{(\omega)}$, and let $\mathcal{R}_\omega$ be the submodule of $\mathcal{F}$ generated by the right action R of G on $\Delta^{(\omega)}$. Then $\mathcal{L}_\omega$ and $\mathcal{R}_\omega$ are irreducible highest weight representations of G with respect to the actions L and R with highest weight ω. Similarly let $\mathcal{I}^{(\omega)}$ be the sub-bimodule of $\mathcal{F}$ generated by the joint action $L \bigodot R$ on $\Delta^{(\omega)}$. Then $\mathcal{I}^{(\omega)}$ is an irreducible highest weight representation of the joint action $L \bigodot R$ with highest weight $(\omega|\omega)$.

One checks from the definitions that products of determinant functions are $\underleftarrow{\text{orthogonal}}$ whenever any of the terms are $\underleftarrow{\text{orthogonal}}$. For example,

$$\Delta^{I_1}_{J_1} \Delta^{I_2}_{J_2} \qquad \text{and} \qquad \Delta^{I'_1}_{J'_1} \Delta^{I'_2}_{J'_2}$$

are $\underleftarrow{\text{orthogonal}}$ if any of $I_1 \neq I'_1$, $J_1 \neq J'_1$ etc. From this and proposition 4.10 we have the following.

Corollary 4.14. $\mathcal{I}^{(\omega)}$ and $\mathcal{I}^{(\omega')}$ are $\underleftarrow{\text{orthogonal}}$ if $\omega \neq \omega'$.

We further generalize our situation with the following theorem.

Theorem 4.15. Let

$$\Omega = \sum_\omega \oplus \left(F^{(\omega)}(V) \bigotimes F^{(\omega)}(V^t) \right)$$

be as in (6). Then the map in proposition 4.12 extends to the map

$$\Phi \odot \Psi : \Omega \longrightarrow \mathcal{F}$$

defined by

$$\Phi \odot \Psi = \sum_\omega (\Phi^{(\omega)} \odot \Psi^{(\omega)}).$$

Thus the image

$$\Phi \odot \Psi(\Omega) = \sum_\omega \oplus \mathcal{I}^{(\omega)}$$

is a multiplicity-free $\underleftarrow{\text{orthogonal}}$ algebraic direct sum of irreducible highest weight submodules.

4.3. Decomposition of $\mathcal{F}$

Theorem 4.16. Let $\mathcal{P}$ be as in section 3.2. Then the function $\Phi \odot \Psi$ maps Ω onto $\mathcal{P}$.

Thus

$$\mathcal{P} = \sum_{\omega} \oplus \mathcal{I}^{(\omega)}$$

where the orthogonal algebraic direct sum is taken over distinct highest weights ω.

Proof. From the classical case [20] we have, for any $s \leqslant t$, that the space of polynomials $\mathcal{P}_{s,t}$ decomposes into an orthogonal algebraic direct sum

$$\mathcal{P}_{s,t} = \sum_{\omega} \oplus \mathcal{I}_{st}^{(\omega)} \tag{16}$$

where each $\mathcal{I}_{st}^{(\omega)}$ is the submodule of $\mathcal{P}_{s,t}$ generated by the action $L \odot R$ restricted to $GL(s,t) \times GL(s,t)$ on the highest weight vector $\Delta_{st}^{(\omega)}$ (the restriction of $\Delta^{(\omega)}$ to G_{st}^{aug}), i.e.

$$\mathcal{I}_{st}^{(\omega)} = \{\Delta_{A^t B}^{(\omega)} | A, B \in GL(s,t)\}$$

where, as in section 4.1,

$$\Delta_{A^t B}^{(\omega)}(z) := \Delta^{(\omega)}(A^t z B) \qquad \text{for all } z \in G_{st}^{\mathrm{aug}}$$

and the sum in (16) is taken over all

$$\omega = \sum_{i=1}^{n} k_i \omega_{\underline{M}_i}$$

with $s \leqslant m_i \leqslant t$†. (In what follows we will describe this situation by writing $s \leqslant \omega \leqslant t$.)

Now certainly $\mathcal{I}_{st}^{(\omega)} \subseteq \mathcal{I}^{(\omega)}$, and since

$$\Phi^{(\omega)} \odot \Psi^{(\omega)}(F^{(\omega)}(V) \otimes F^{(\omega)}(V^t)) = \mathcal{I}^{(\omega)}$$

we have that the intertwining map

$$\Phi^{(\omega)} \odot \Psi^{(\omega)} : F^{(\omega)}(V) \otimes F^{(\omega)}(V^t) \longrightarrow \mathcal{I}_{st}^{(\omega)}$$

is onto for any $s \leqslant t$.

Since

$$\mathcal{P} = \lim_{\substack{s \longrightarrow -\infty \\ t \longrightarrow +\infty}} \mathcal{P}_{st}$$

the result follows. □

Since $\mathcal{F} = \overline{\mathcal{P}}$ we have

$$\mathcal{F} = \sum_{\omega} \hat{\oplus} \mathcal{I}^{(\omega)}$$

where the hat indicates closure of the algebraic direct sum, and so the sum can be ‘infinite’, a notion we now make precise.

† Here the classical signature $\alpha = (\alpha_1, \ldots, \alpha_n)$ with

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n \geqslant 0$$

corresponds to the highest weight $\omega = (\ldots, \omega_{s-1}, \omega_s, \ldots, \omega_t, \omega_{t+1}, \ldots)$ with

$$\ldots = \omega_{s-1} = \omega_s \geqslant \omega_{s+1} \geqslant \cdots \geqslant \omega_t = \omega_{t+1} = \ldots \geqslant 0.$$

Theorem 4.17. Any $f \in \mathcal{F}$ can be written as a series of orthogonal terms

$$f = \sum_\omega f^{(\omega)}$$

where $f^{(\omega)} \in \mathcal{I}^{(\omega)}$, which converges in the topology of the projective limit.

Proof. Let $f \in \mathcal{F}$, then for any $s' \leqslant s \leqslant t \leqslant t'$ we know that there is a sequence $\{f_{s't'}^{(\omega)}\}_\omega$ in $\mathcal{F}_{s',t'}$ such that

$$f_{s',t'} = \sum_\omega f_{s't'}^{(\omega)}$$

and which restricts to

$$f_{st} = \sum_\omega f_{st}^{(\omega)}.$$

Since $\mathcal{F} = \varprojlim \mathcal{F}_{s,t}$, there exists a unique sequence $\{f^{(\omega)}\} \in \mathcal{F}$ that restricts to $\{f_{st}^{(\omega)}\}$ for all $s \leqslant t$. Since

$$f_{st} = \sum_\omega f_{st}^{(\omega)} \qquad \text{for all } s \leqslant t$$

we have that

$$f = \sum_\omega f^{(\omega)}$$

in the topology of the projective limit. That is, if $\{\omega\}$ denotes the collection of all highest weights, then the map $\{\omega\} \longrightarrow \mathcal{F}$ completes the following commutative diagram.

$$\begin{tikzcd}
f = \sum_\omega f^{(\omega)} \arrow[dr, "\mathrm{Res}_{s't'}"] \arrow[ddr, "\mathrm{Res}_{st}"'] & & \{\omega\} \arrow[ll] \arrow[dl] \arrow[ddl] \\
& f_{s't'} = \sum_\omega f_{s't'}^{(\omega)} \arrow[d] & \\
& f_{st} = \sum_\omega f_{st}^{(\omega)} &
\end{tikzcd}$$

$\square$

4.4. Representations in $\mathcal{F} \otimes \mathcal{F}$

As in section 4.2, consider the (exterior) tensor product representation $L \otimes R$ of $G \times G$ on $\mathcal{F} \otimes \mathcal{F}$ given by

$$L \otimes R(A, B)(f, g)(Z) = L(A)f(Z) \otimes R(B)g(Z).$$

Since the map $\Phi^{(m)} : F^{(m)}(V) \longrightarrow \mathcal{F}$ intertwines $\tilde{L}$ and L, and the map $\Psi^{(m)} : F^{(m)}(V^t) \longrightarrow \mathcal{F}$ intertwines $\tilde{R}$ and R, the map

$$\Phi^{(m)} \otimes \Psi^{(m)} : F^{(m)}(V) \otimes F^{(m)}(V^t) \longrightarrow \mathcal{F} \otimes \mathcal{F}$$

defined by

$$\nu_I \otimes \nu_J^t \longmapsto \Delta_I^M \otimes \Delta_{\underline{M}}^J$$

intertwines $\tilde{L} \otimes \tilde{R}$ and $L \otimes R$. Thus, the action $L \otimes R$ of $G \times G$ on the highest weight vector $\Delta_{\underline{M}}^{\underline{M}} \otimes \Delta_{\underline{M}}^{\underline{M}}$ generates $\mathcal{L}_{\omega_{\underline{M}}} \otimes \mathcal{R}_{\omega_{\underline{M}}}$, which is then an irreducible highest weight submodule of $\mathcal{F} \otimes \mathcal{F}$ with highest weight $(\omega_{\underline{M}}|\omega_{\underline{M}})$. But the map $\Phi^{(m)} \odot \Psi^{(m)}$ of theorem 4.12 also intertwines $\tilde{L} \otimes \tilde{R}$ and $L \odot R$ and so we must have that $\mathcal{I}^{(\omega_{\underline{M}})}$ and $\mathcal{L}_{\omega_{\underline{M}}} \otimes \mathcal{R}_{\omega_{\underline{M}}}$ are isomorphic as $G \times G$ modules. This motivates the following.

Proposition 4.18. Let ω, $\Phi^{(\omega)}$, $\Psi^{(\omega)}$ be as in corollary 4.12. Then the map

$$\Phi^{(\omega)} \otimes \Psi^{(\omega)} : F^{(m_1)}(V) \otimes \cdots \otimes F^{(m_n)}(V) \bigotimes F^{(m_1)}(V^t) \otimes \cdots \otimes F^{(m_n)}(V^t) \longrightarrow \mathcal{F} \otimes \mathcal{F}$$

defined by

$$\nu_{I_1} \otimes \cdots \otimes \nu_{I_n} \bigotimes \nu_{J_1}^t \otimes \cdots \otimes \nu_{J_n}^t \longmapsto \Delta_{I_1}^{M_1} \ldots \Delta_{I_n}^{M_n} \otimes \Delta_{\underline{M}_1}^{J_1} \ldots \Delta_{\underline{M}_n}^{J_1}$$

intertwines $\tilde{L} \bigotimes \tilde{R}$ and $L \bigotimes R$. Therefore, $\mathcal{L}_\omega \otimes \mathcal{R}_\omega$ and $\mathcal{I}^{(\omega)}$ are isomorphic $G \times G$ modules.

Remark 4.19. It is easy to see that the above isomorphism is given directly by

$$\Delta_I^M \otimes \Delta_{\underline{M}}^J \longmapsto \Delta_I^J.$$

4.5. Structure of irreducible modules

Returning to the simple case, where $\omega_{\underline{M}}$ is a fundamental weight, consider the submodule $\mathcal{L}_{\omega_{\underline{M}}} \otimes \mathcal{R}_{\omega_{\underline{M}}}$ of $\mathcal{F} \otimes \mathcal{F}$ which is generated by the action $L \otimes R$ of $G \times G$ on the highest weight vector $\Delta_{\underline{M}}^{M} \otimes \Delta_{\underline{M}}^{M}$. Restricting the action $L \otimes R$ of $G \times G$ to the subgroup $G \times Id$ we obtain a left submodule $[L(G)\Delta_{\underline{M}}^{M}] \otimes \Delta_{\underline{M}}^{M}$ of $\mathcal{L}_{\omega_{\underline{M}}} \otimes \mathcal{R}_{\omega_{\underline{M}}}$ isomorphic to $\mathcal{L}_{\omega_{\underline{M}}}$. Moreover, as I varies over shuffles of $\underline{M}$, the action of $G \times Id$ on the vectors $\Delta_{\underline{M}}^{M} \otimes \Delta_{\underline{M}}^{I}$ generates distinct copies of $\mathcal{L}_{\omega_{\underline{M}}}$ in $\mathcal{L}_{\omega_{\underline{M}}} \otimes \mathcal{R}_{\omega_{\underline{M}}}$, i.e. the submodules $[L(G)\Delta_{\underline{M}}^{M}] \otimes \Delta_{\underline{M}}^{I}$. Since $\Delta_{\underline{M}}^{I} = \Psi^{(m)}(\nu_I^t)$ and the ν_I^t form a basis of $F^{(m)}(V^t)$, the functions $\Delta_{\underline{M}}^{I}$ form a basis of $\mathcal{R}_{\omega_{\underline{M}}}$ and we have

(1) $\mathcal{L}_{\omega_{\underline{M}}} \otimes \mathcal{R}_{\omega_{\underline{M}}}$ is a direct sum of copies of $\mathcal{L}_{\omega_{\underline{M}}}$, and also a direct sum of copies of $\mathcal{R}_{\omega_{\underline{M}}}$. These copies are indexed by the set $\{I | I \text{ is a shuffle of } \underline{M}\}$.

(2) By remark 4.19 we have that $\mathcal{I}_{\omega_{\underline{M}}}$ is isomorphic to a direct sum of copies of $\mathcal{L}_{\omega_{\underline{M}}}$. Each copy is generated by the left action L of G on the function $\Delta_{\underline{M}}^{I}$ as I ranges over all shuffles of $\underline{M}$.

(3) Similarly, $\mathcal{I}_{\omega_{\underline{M}}}$ is isomorphic to a direct sum of copies of $\mathcal{R}_{\omega_{\underline{M}}}$, each generated by the right action R of G on the function Δ_I^{M}.

One could say that $\omega_{\underline{M}}$ is the 'lowest highest weight'.

We generalize as follows.

Proposition 4.20.

(1) For

$$\omega = \sum_{i=1}^{n} \omega_{\underline{M}_i}$$

(where some of the $\underline{M}_i$ may be repeated) the irreducible $G \times G$ submodule $\mathcal{L}_\omega \otimes \mathcal{R}_\omega$ of $\mathcal{F} \otimes \mathcal{F}$ is a direct sum of irreducible left submodules, each isomorphic to $\mathcal{L}_\omega$ and each generated by the action $L \otimes R$ of $G \times Id$ on the vector

$$\Delta^{\underline{M}_1}_{\underline{M}_1} \ldots \Delta^{\underline{M}_n}_{\underline{M}_n} \otimes \Delta^{I_1}_{\underline{M}_1} \ldots \Delta^{I_n}_{\underline{M}_n}$$

where I_1 is a shuffle of $\underline{M}_1$, ..., I_n is a shuffle of $\underline{M}_n$.

(2) The irreducible $G \times G$ submodule $\mathcal{I}^{(\omega)}$ of $\mathcal{F}$ is isomorphic to a direct sum of irreducible left G-modules, each a copy of $\mathcal{L}_\omega$, and each generated by the left action L of G on the vector

$$\Delta^{I_1}_{\underline{M}_1} \ldots \Delta^{I_n}_{\underline{M}_n}.$$

Similarly, $\mathcal{I}^{(\omega)}$ is isomorphic to a direct sum of right G-modules, each isomorphic to $\mathcal{R}_\omega$, and each generated by the right action R on the vector

$$\Delta^{\underline{M}_1}_{I_1} \ldots \Delta^{\underline{M}_n}_{I_n}.$$

From this and the remarks following corollary 4.13 we have the following.

Corollary 4.21. Let $\mathcal{L}_{I_1,\ldots,I_n}$ be the irreducible left submodule generated by the vector

$$\Delta^{I_1}_{\underline{M}_1} \ldots \Delta^{I_n}_{\underline{M}_n}$$

and let $\mathcal{L}_{I'_1,\ldots,I'_n}$ be the irreducible left submodule generated by the vector

$$\Delta^{I'_1}_{\underline{M}_1} \ldots \Delta^{I'_n}_{\underline{M}_n}.$$

Then if $f \in \mathcal{L}_{I_1,\ldots,I_n}$ and if $f' \in \mathcal{L}_{I'_1,\ldots,I'_n}$ then f and f' are orthogonal.

$\longleftarrow$

Now suppose that $\mathcal{L}'$ is any irreducible left submodule of $\mathcal{F}$. Since $\mathcal{F} = \sum \hat{\oplus} \mathcal{I}^{(\omega)}$, $\mathcal{L}'$ must intersect some $\mathcal{I}^{(\omega)}$ non-trivially. But the intersection of left submodules is a left submodule, and each $\mathcal{I}^{(\omega)}$ is a direct sum of irreducible left submodules, so $\mathcal{L}'$ must be a direct summand of $\mathcal{I}^{(\omega)}$ for some ω. Therefore we obtain the following.

Proposition 4.22. For each highest weight ω, $\mathcal{I}^{(\omega)}$ is the *isotypic component* characterized by ω. That is, each $\mathcal{I}^{(\omega)}$ is the direct sum of *all* irreducible left submodules isomorphic to $\mathcal{L}_\omega$. Similarly, each $\mathcal{I}^{(\omega)}$ is the direct sum of all irreducible right submodules isomorphic to $\mathcal{R}_\omega$.

4.6. *Characterization of highest weight modules*

For any $l \in \mathbb{Z}$ let $\beta = (\ldots, \beta_l, \beta_{l+1}, \beta_{l+2}, \ldots)$ be a sequence of non-negative integers with $\cdots \geqslant \beta_l \geqslant \beta_{l+1} \geqslant \beta_{l+2} \geqslant \cdots$ (an example of particular interest is when β is a highest weight). Let $\mathcal{B}$ denote the Borel subgroup of upper triangular matrices in GL_∞. That is

$$\mathcal{B} = \{B \in G | B_{i,j} = 0 \text{ for } i > j\}.$$

For each β we define a holomorphic character $\pi^{(\beta)} : \mathcal{B} \longrightarrow \mathbb{C}^*$ (the non-zero complex numbers) by

$$\pi^{(\beta)}(B) = \prod_{i \in \mathbb{Z}} B_{ii}^{\beta_i} \qquad \text{for all } B \in \mathcal{B}.$$

Note that $\pi^{(\beta)}(B)$ is well defined for each $B \in \mathcal{B}$. We say a function $f \in \mathcal{F}$ transforms *R-covariantly with respect to* $\mathcal{B}$ *with signature* β if

$$R(B)f(Z) = \pi^{(\beta)}(B)f(Z) \qquad \text{for all } B \in \mathcal{B} \text{ and } Z \in G^{\text{aug}}.$$

Clearly the set of all such functions is a subspace of $\mathcal{F}$, denoted by $V^{(\beta)}$, which is invariant under left translation by G. Via a routine application of lemma 3.6 we see that if $\Delta^{(\omega)}$ is a highest weight vector, then $\Delta^{(\omega)} \in V^{(\omega)}$, and so $\mathcal{L}_\omega \subset V^{(\omega)}$. In fact, if we define $\mathcal{L}_{\omega st}$ and $V_{st}^{(\omega)}$ in the obvious way, that is

$\mathcal{L}_{\omega st}$ = left submodule generated by the action of $GL(s,t)$ on $\Delta_{st}^{(\omega)}$ for $s \leqslant \omega \leqslant t$

$V_{st}^{(\omega)}$ = subspace of $\mathcal{F}_{s,t}$ that transforms R-covariantly with respect to

$$\mathcal{B}_{st} = \mathcal{B} \bigcap GL(s,t) \text{ with signature } \omega$$

then by the Borel–Weil theorem, the two coincide. Taking the limit as $s \longrightarrow -\infty$ and $t \longrightarrow \infty$ we have the following.

Proposition 4.23. The highest weight left (respectively right) submodule of $\mathcal{F}$ with highest weight ω is characterized by transforming R-covariantly (respectively L-covariantly) with respect to the Borel subgroup, with signature ω.

4.7. Representations of GL_∞ in $\mathfrak{F}$ and decomposition of $\mathfrak{F}$

Here we discuss our previous results in the more familiar context of an inductive limit of Hilbert spaces.

Restricting the actions L and R of GL_∞ on $\mathcal{F}$ yields the classical left and right actions we call L_{st} and R_{st} of $GL(s,t)$ on $\mathcal{F}_{s,t}$. If $\omega = \sum k_i \omega_{\underline{M}_i}$ as before we use the notation $s \leqslant \omega \leqslant t$ to denote the situation where $s \leqslant m_i \leqslant t$ in the above sum. As in the proof of theorem 4.16, let $\mathcal{I}_{st}^{(\omega)}$ be the sub $GL(s,t) \times GL(s,t)$ bimodule generated by the action $L_{st} \odot R_{st}$ on $\Delta_{st}^{(\omega)}$ with $s \leqslant \omega \leqslant t$. Then it is well known that $\mathcal{I}_{st}^{(\omega)}$ is an irreducible representation of $GL(s,t) \times GL(s,t)$. We also denote by $\mathcal{L}_{\omega st}$ and $\mathcal{R}_{\omega st}$ the irreducible submodules of $\mathcal{F}_{s,t}$ generated by the left and right actions on the highest weight vector $\Delta_{st}^{(\omega)}$.

Now for $s' \leqslant s \leqslant t \leqslant t'$ the action L_{st} (or R_{st}) of $GL(s,t)$ on $\mathcal{F}_{s,t}$ commutes with the isometric embedding

$$\text{Inc}_{st}^{s't'} : \mathcal{F}_{s,t} \hookrightarrow \mathcal{F}_{s',t'}$$

and so from the properties of the inductive limit there arises in $\mathfrak{F}$ a representation of GL_∞ that we call $\underset{\longrightarrow}{L}$ (or $\underset{\longrightarrow}{R}$), uniquely defined by

$$\underset{\longrightarrow}{L}(g)f = L_{st}(g)f \qquad \text{whenever } g \in GL(s,t) \text{ and } f \in \mathcal{F}_{s,t}$$

and we construct the representations $\underset{\longrightarrow}{L} \odot \underset{\longrightarrow}{R}$ and $\underset{\longrightarrow}{L} \otimes \underset{\longrightarrow}{R}$ in the obvious way.

Set

$$\mathfrak{I}^{(\omega)} = \underset{\longrightarrow}{\text{Ind}}\, \mathcal{I}_{st}^{(\omega)}.$$

Since each $\mathcal{I}_{st}^{(\omega)}$ is a closed subspace of $\mathcal{F}_{s,t}$ (being finite dimensional), $\mathfrak{I}^{(\omega)}$ is a closed subspace of $\mathfrak{F}$ in the inductive limit topology induced by the norm. Clearly $\mathcal{I}_{st}^{(\omega)}$ and $\mathcal{I}_{st}^{(\omega')}$ are orthogonal whenever $\omega \neq \omega'$, so in this case $\mathfrak{I}^{(\omega)}$ and $\mathfrak{I}^{(\omega)'}$ are also orthogonal. In an identical fashion we define

$$\mathfrak{L}_\omega = \underset{\longrightarrow}{\text{Ind}}\, \mathcal{L}_{\omega st}$$

$$\mathfrak{R}_\omega = \underset{\longrightarrow}{\text{Ind}}\, \mathcal{R}_{\omega st}.$$

Theorem 4.24. Let $\mathfrak{I}^{(\omega)}$ be the inductive limit $\underset{\longrightarrow}{\mathrm{Ind}}\,\mathcal{I}_{st}^{(\omega)}$. Then the representation $\underset{\longrightarrow}{L} \odot \underset{\longrightarrow}{R}$ of $GL_\infty \times GL_\infty$ on $\mathfrak{I}^{(\omega)}$ is irreducible.

Proof. We first adapt a result that we will quote from Dixmier [4].

Theorem 4.25. Let $\mathcal{A}$ be an algebra with an involution, let $\mathcal{H}$ be a Hilbert space, and let π be a representation of $\mathcal{A}$ in $\mathcal{H}$. Let $\mathcal{L}(\mathcal{H})$ denote the space of linear operators on $\mathcal{H}$. Then the following are equivalent;

(i) the only closed subspaces of $\mathcal{H}$ which are π invariant are $\{0\}$ and $\mathcal{H}$.
(ii) The only elements of $\mathcal{L}(\mathcal{H})$ which commute with $\pi(\mathcal{A})$ are the scalars.
(iii) Every vector in $\mathcal{H}$ is a cyclic vector.

Now let $\mathcal{L}(\mathfrak{I}^{(\omega)})$ be the space of linear operators on $\mathfrak{I}^{(\omega)}$, and suppose that $\mathcal{O} \in \mathcal{L}(\mathfrak{I}^{(\omega)})$ commutes with the operators $L \odot R(A, B)$ for all $A, B, \in GL_\infty$. Let $\mathrm{proj}_{st} : \mathfrak{I}^{(\omega)} \longrightarrow \mathcal{I}_{st}^{(\omega)}$ be the projection operator of $\mathfrak{I}^{(\omega)}$ onto its subspace $\mathcal{I}_{st}^{(\omega)}$. Then $\mathrm{proj}_{st}\mathcal{O}$ commutes with the operators $\underset{\longrightarrow}{L} \odot \underset{\longrightarrow}{R}\,(A, B)$ (for all $A, B, \in GL(s,t)$) in $\mathcal{L}(\mathcal{I}_{st}^{(\omega)})$, and since each $\mathcal{I}_{st}^{(\omega)}$ is irreducible, by the above theorem 4.25, $\mathrm{proj}_{st}\mathcal{O}$ must be a scalar c_{st}. But whenever $s' \leqslant s$ and $t' \geqslant t$, each $\mathcal{F}_{s,t}$ embeds isometrically into $\mathcal{F}_{s',t'}$, so c_{st} must be the *same* scalar c for all $s \leqslant t$, and hence the only operator in $\mathcal{L}(\mathfrak{I}^{(\omega)})$ that commutes with $\mathcal{O}$ is the scalar operator c. So by theorem 4.25 again, the space $\mathfrak{I}^{(\omega)}$ must be irreducible. □

By an identical argument we have the following.

Proposition 4.26. The actions $\underset{\longrightarrow}{L}$ and $\underset{\longrightarrow}{R}$ of GL_∞ on the submodules $\mathfrak{L}_\omega$ and $\mathfrak{R}_\omega$ are irreducible.

By a routine application of the above techniques we have the following.

Theorem 4.27. Let $V_{st}^{(\omega)}$be as in section 4.6, let $\mathfrak{V}^{(\omega)} = \underset{\longrightarrow}{\mathrm{Ind}} V_{st}^{(\omega)}$. Then $\mathfrak{V}^{(\omega)}$ transforms $\underset{\longrightarrow}{R}$-covariantly with respect to the Borel subgroup with signature ω, and $\mathfrak{V}^{(\omega)} = \mathfrak{L}_\omega$.

We have noted before that $\mathcal{P}_{s,t}$ is the orthogonal algebraic direct sum

$$\mathcal{P}_{s,t} = \sum_{s \leqslant \omega \leqslant t} \oplus \mathcal{I}_{st}^{(\omega)} \qquad \text{for all } s \leqslant t \tag{17}$$

so that $\mathcal{F}_{s,t}$ is the Hilbert space direct sum

$$\mathcal{F}_{s,t} = \sum_{s \leqslant \omega \leqslant t} \hat{\oplus} \mathcal{I}_{st}^{(\omega)} \qquad \text{for all } s \leqslant t.$$

We next obtain a similar decomposition for the inductive limit $\mathfrak{F}$.

Theorem 4.28. Let $\mathfrak{F}$ be the inductive limit $\underset{\longrightarrow}{\mathrm{Ind}}\,\mathcal{F}_{s,t}$, let $\mathfrak{I}^{(\omega)}$ be the inductive limit $\underset{\longrightarrow}{\mathrm{Ind}}\,\mathcal{I}_{st}^{(\omega)}$, then

$$\mathfrak{F} = \sum_{\omega} \hat{\oplus} \mathfrak{I}^{(\omega)}$$

where the Hilbert space direct sum is taken over distinct highest weights ω, and each $\mathfrak{I}^{(\omega)}$ is an irreducible $GL_\infty \times GL_\infty$ module with respect to the joint action $L \odot R$.

Proof. We first note that the sum (17) is an algebraic direct sum, so we may interchange the summation and inductive limit, that is

$$\begin{aligned}\mathfrak{P} &= \underrightarrow{\mathrm{Ind}}\, \mathcal{P}_{s,t} \\ &= \underrightarrow{\mathrm{Ind}} \left(\sum \oplus \mathcal{I}_{st}^{(\omega)} \right) \\ &= \sum \oplus \underrightarrow{\mathrm{Ind}}\, \mathcal{I}_{st}^{(\omega)} \\ &= \sum \oplus \mathfrak{I}^{(\omega)}.\end{aligned}$$

Thus,

$$\mathfrak{F} = \overline{\mathfrak{P}} = \sum_{\omega} \hat{\oplus} \mathfrak{I}^{(\omega)}.$$

□

Finally, we note that we can realize the notion of highest weight vectors in $\mathfrak{F}$ by the abstract symbol $\Delta^{(\omega)}$ in the sense that, for any $s \leqslant \omega \leqslant t$ the restriction of $\Delta^{(\omega)}$ to $\mathcal{F}_{s,t}$ is a highest weight vector with highest weight ω, even though the function $\Delta^{(\omega)}$ itself is not in $\mathfrak{F}$. By theorem 4.25, any such function generates an irreducible representation.

4.8. Concluding remarks

(1) In this paper we have realized all irreducible highest weight representations of GL_∞ on a projective limit of Fock spaces. The representations on this space seem the most natural generalizations of the finite-dimensional case, in the sense that highest weights and highest weight vectors have clear generalizations, and the restriction maps give us back the finite-dimensional cases. In this context, we also discuss representations on the inductive limits of Fock spaces, which similarly project onto the finite-dimensional cases. We further note that we could induce a Hilbert space structure on $\mathcal{F}$ from the wedge space via the intertwining map, but this does not seem very fruitful.

(2) If

$$U_\infty = \{A \in GL_\infty | AA^\dagger = Id\}$$

then everything proven about GL_∞ is also true for U_∞ via the standard argument of Weyl's 'unitarian trick'.

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